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1995 J. Phys. A: Math. Gen. 28 5287

(http://iopscience.iop.org/0305-4470/28/18/017)

J. Phys. A: Math. Gen. 28 (1995) 5287–5289. Printed in the UK

# Smoothed Wigner function of a quantum damped oscillator

E A Akhundova† and M A Mukhtarov‡

† Institute of Physics, Academy of Sciences, pr. H Gavid 33, Baku 370143, Republic of Azerbaijan

‡ Joint Institute for Nuclear Research, Laboratory of Theoretical Physics, Moscow Region, Dubna 141980, Russia

Received 21 February 1995

**Abstract.** The non-stationary quadratic quantum system which can be considered as a quantum model of a damped oscillator is investigated in the framework of the Wigner representation. The explicit expressions of the ordinary and smoothed Wigner functions for this system are obtained.

In the last few years the Wigner–Weyl representation [1,2] has played a very important role in the investigation of modern quantum mechanical problems [3–10]. For instance, with the use of the Wigner function a quantum correction to the classical results can be obtained [3]. This representation has served as a very convenient tool for studying quantum mechanical systems with quadratic Hamiltonians with respect to the coordinate and momentum operators [11–15]. The Wigner function allows one to find an average value of an arbitrary physical quantity, but it cannot be interpreted as a probability because it can take negative values. By smoothing the Wigner function with different weight functions, one can ensure its positive definiteness, and the resulting function is known as the smoothed Wigner function [17–19]. The smoothed Wigner function leads to errors in calculations of average values compared with exact quantum mechanical results, but in the classical limit it begins to play an essential role.

In this work, using the results [14–16], we consider in the framework of the Wigner representation a quantum system with the Hermitian non-stationary Hamiltonian

$$\hat{H}(t) = \tfrac{1}{2}\left(\hat{p}^2 e^{-2\Gamma(t)} + \omega_0^2(t)\, e^{2\Gamma(t)}\hat{x}^2\right) - f(t)\, e^{2\Gamma(t)}\hat{x} \tag{1}$$

with the corresponding equation of motion

$$\begin{aligned} \dot{x} &= p e^{-2\Gamma(t)} \\ \dot{p} &= -\omega_0^2(t)\, e^{2\Gamma(t)} x + f(t)\, e^{2\Gamma(t)} \\ \ddot{x} &+ 2\dot{\Gamma}(t)\,\dot{x} + \omega_0^2(t)\, x = f(t)\,. \end{aligned} \tag{2}$$

As can be seen from (1) and (2), the quantum system with this type of Hamiltonian can be considered as a quantum analogue of the classical damped forced harmonic oscillator with the time-dependent parameters. This equation was considered earlier in many papers. For more details on the choice of Hamiltonian and the problem of initial conditions for it, see [16] and references therein.

Let us find the integrals of motion of the system, i.e. the operators $\hat{I}(t)$ satisfying the equation

$$\left[\mathrm{i}\,(\partial/\partial t) - \hat{H}\,,\, \hat{I}\right] = 0 \qquad (\hbar = 1)$$

0305-4470/95/185287+03$19.50  

For one-dimensional quadratic Hamiltonian systems all the integrals of motion can be constructed from two independent linear ones. For the present system one has two mutually Hermitian conjugate linear integrals of motion satisfying

$$\left[\hat{A}(t)\,,\,\hat{A}^{+}(t)\right]=1.$$

The operator $\hat{A}(t)$ has the form

$$\begin{aligned}\hat{A}(t)&=\frac{\mathrm{i}}{\sqrt{2}}\left(\epsilon(t)\,\hat{p}-\dot{\epsilon}(t)\,\mathrm{e}^{2\Gamma(t)}\hat{x}\right)+\frac{\delta(t)}{\sqrt{2}}\\ \delta(t)&=-\mathrm{i}\int\epsilon(\tau)\,\mathrm{e}^{2\Gamma(t)}f(t)\ \mathrm{d}\tau\end{aligned}\tag{3}$$

where $\epsilon(t)$ is a complex function satisfying the equation

$$\ddot{\epsilon}+2\dot{\Gamma}\dot{\epsilon}+\omega_0^2(t)\,\epsilon=0\tag{4}$$

and the additional relation

$$\mathrm{e}^{2\Gamma(t)}\left(\dot{\epsilon}\epsilon^*+\dot{\epsilon}^*\epsilon\right)=2\mathrm{i}.\tag{5}$$

Let us consider an operator of the following form:

$$\hat{K}(t)=\hat{A}\hat{A}^{+}+\tfrac{1}{2}.$$

Using equations (3)–(5) it is easy to check that this operator is also an integral of motion, and it has the meaning of a quasi-particle number operator. The eigenfunctions of this operator satisfy the Schrödinger equation of the system

$$\hat{K}\psi_n=\left(n+\tfrac{1}{2}\right)\psi_n.$$

The eigenfunctions $\psi_n$ have the form

$$\begin{aligned}\psi_n(x,t)=(n!)^{-\frac{1}{2}}\left(\frac{\epsilon^*}{2\epsilon}\right)^{\frac{n}{2}}\left(\pi\epsilon^2\right)^{-\frac{1}{4}}\exp\Bigg[&\frac{\mathrm{i}\dot{\epsilon}}{2\epsilon}\mathrm{e}^{2\Gamma(t)}x^2-\frac{x\delta}{\epsilon}-\frac{\epsilon^*}{4\epsilon}\delta^*-\tfrac{1}{4}|\delta|^2\\ &-\frac{\mathrm{i}}{2}\int\mathrm{Im}\left(\delta\dot{\delta}^*\right)\ \mathrm{d}\tau\Bigg]H_n\left(\frac{x+\mathrm{Re}\,(\epsilon^*\delta)}{|\epsilon|}\right)\end{aligned}$$

where the $H_n(x)$ are Hermite polynomials.

The corresponding Wigner function is as follows:

$$\begin{aligned}W_n(p,q)&=2\,(-1)^n\,\mathrm{e}^{-2z(t)}L_n\left(4z(t)\right)\\ z(t)=\tfrac{1}{2}\big[|\epsilon|^2p^2+|\dot{\epsilon}|^2\mathrm{e}^{4\Gamma}x^2&-2\mathrm{e}^{2\Gamma}\,\mathrm{Re}\left(\dot{\epsilon}\epsilon^*\right)xp+\mathrm{Im}\left(\epsilon^*\delta\right)p-\mathrm{e}^{2\Gamma}\,\mathrm{Im}\left(\dot{\epsilon}^*\delta\right)x+|\delta|^2\\ &+\mathrm{i}\mathrm{e}^{2\Gamma}\,\mathrm{Re}\left(\dot{\epsilon}\epsilon^*\right)\big]\end{aligned}$$

where the $L_n(x)$ are Laguerre polynomials.

To determine the smoothed Wigner function it is necessary to consider the 'probability' $U_{\Delta p\Delta q}$ of the phase-point hit in the finite domain $\Delta p\Delta q$ of the phase plane:

$$U_{\Delta p\Delta q}=\int\!\!\int_{\Delta p\Delta q}W\left(p+p_1,q+q_1\right)\ \mathrm{d}p_1\ \mathrm{d}q_1.$$

Instead of the exact boundary domain one considers the 'spread' boundary domain by introducing the following quantity:

$$U_{\Delta p\Delta q}=\iint_{-\infty}^{\infty}\exp\left(-\frac{p_1^2}{2(\Delta p)^2}-\frac{q_1^2}{2(\Delta q)^2}\right)W\left(p+p_1,q+q_1\right)\,\mathrm{d}p_1\,\mathrm{d}q_1\,.$$

This method of obtaining of the smoothed Wigner function is given in more detail in [18, 19].

Finally, for the smoothed Wigner function we obtain (assuming that $\triangle p \Delta q = \frac{1}{2}$)

$$\bar{W}_n(p,q) = \frac{1}{\pi}\left(U_{\triangle p \Delta q}\right)_n = \left(2^n n! \pi^{\frac{1}{2}} \Delta q |\epsilon|\right)^{-1}$$
$$\times \exp\left[-\frac{\operatorname{Re}(\epsilon\delta^*)}{2|\epsilon|^2} - \tfrac{1}{2}|\delta|^2 - \mathrm{i}\int \operatorname{Im}\left(\delta\dot{\delta}^*\right)\,\mathrm{d}\tau - \frac{q}{4(\Delta q)^2} + \operatorname{Re}\left(L^2 S^{-1}\right)\right]$$
$$\times |S| \frac{1}{|\epsilon|^n}\left[\left(|\epsilon|^2 - S^{-1}\right)\left(|\epsilon|^2 - \left(S^*\right)^{-1}\right)\right]^{\frac{n}{2}} \left| H_n\left[\frac{\operatorname{Re}(\epsilon^*\delta) + LS^{-1}}{\left(|\epsilon|^2 - S^{-1}\right)^{\frac{1}{2}}}\right]\right|^2$$

where

$$L = \frac{q^2}{2(\Delta q)^2} - \frac{\delta}{\epsilon} - \mathrm{i}p \qquad S = \frac{1}{2(\Delta q)^2} - \frac{\mathrm{i}\dot{\epsilon}}{\epsilon}\mathrm{e}^{2\Gamma(t)}.$$

## Acknowledgments

We are very grateful to Professors V I Man'ko and V V Dodonov for numerous discussions and suggestions. MAM acknowledges partial financial support under INTAS grant No 9.3-127.

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